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Is the principle of equivalence a principle?

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Abstract

The work argues the principle of equivalence to be a theorem and not a principle (in the sense of an axiom). It contains a detailed analysis of the concepts of normal and intertial frame of reference. The equivalence principle is proved to be valid (at every point and along every path) in any gravitational theory based on linear connections. Possible generalizations of the equivalence principle are pointed out.

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1. Introduction

The principle of equivalence played an important role at the early stages of development of general relativity [1–5]. Now, despite historical positions, it is often mentioned as a procedure for transferring results from flat space–time(s) to curved one(s) [1, Ch. 16]. Mathematically this is reflected in the minimal coupling principle used to transfer the Lagrangian formalism from flat to curved manifolds by replacing the flat metric with the (pseudo-)Riemannian one and the usual (partial) derivatives with covariant ones [6].

The equivalence principle is almost everywhere considered as a statement that cannot be proved or need not be proved as it is 'evident' from certain positions and whose consequences are 'reasonable enough' to be taken as a true [4,5].

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The present paper asserts the opinion that when the mathematical back-ground of a gravitational theory is chosen, then the (strong) equivalence 'principle' becomes a theorem (true or not) that can be proved. This is in accordance with the conclusions of [3, Section 61]. There is another case when the equivalence principle is used for selecting the mathematical structure of a gravitational theory. In this case it acts primarily as principle (axiom), but after this selection is made, it again becomes a theorem.

In [7] (see also [8, pp. 5, 160]) is recognized the historical role of the equivalence principle in general relativity, but its exact contents and importance are put under question. By our opinion the latter is a consequence of (some of) the indistinct formulations of this principle and the problem is—is it a theorem or an axiom? These problems are solved completely in the present work. That takes off some of Synge's questions. But we do not share his mind that the equivalence principle is not important nowadays. He is right that now general relativity can be formulated without it. But general relativity is compatible (consistent) with the equivalence principle (in a sense that in this theory it is a provable theorem) as it must be because this principle reflects important empirical observation. Besides, the significance of the principle of equivalence arises (maybe implicitly) in any new gravitational theory as only theories compatible with it can survive.

The present investigation concentrates mainly on the mathematical aspects of the equivalence principle. A physical discussion of this principle can be found in [5, see in particular, pp. 133–137; 4, pp. 334–338], or in [9, Sections 8.2, 9.6].

This work is based mainly on [10–13] and is organized as follows. Section 2 is a brief review of the equivalence principle and its mathematical formulation. Section 3 is devoted to some mathematical theorems closely connected to the subject of this article. Physical conclusions from them are made in Section 4. Section 5 contains remarks about possible extensions of the area of validity of the equivalence principle. Appendix A reviews and discusses some terminological problems. Appendix B contains certain results concerning derivations. Appendix C outlines the proofs of propositions used in this work.

2. The equivalence principle from physical and mathematical point of view

Different formulations of the equivalence principle can be found. They state in one or the other form that (at a point) in a suitable frame of reference the laws of special and general relativity coincide: In [1, Ch. 16] it reads: "In any local Lorentz frame at any time and place in the Universe all (non-gravitational) physical laws take their special relativity form". In [2], one finds it as the assertion that at any space–time point in arbitrary gravitational field a "locally inertial coordinate system can be chosen, in which in a sufficiently small neighborhood of the point, the Nature laws will have the same form as in non-accelerated Cartesian coordinate systems". In [6] it states that "locally the properties of special relativistic matter in a non-inertial frame of reference cannot be distinguished from the properties of the same matter in a corresponding gravitational field". In [9, Section 9.6] the equivalence principle is formulated as follows: "at any point all Nature laws, expressed in local Lorentz coordinates, have the same form as in special relativity".

In fact, these are formulations of the *strong* equivalence principle which is discussed, for instance, in [5, Section 5.2] (see also the references therein). The several weak forms of the principle of equivalence are not a subject of the present investigation.

Above, as well as in other 'physical' publications, the concepts 'local' and 'locally' are not well defined from mathematical view-point and often mean an "infinitesimal surrounding of a fixed point of space–time" [6]. Their strict meaning may be at a point, along a path (curve), in a neighborhood, or on another submanifold (or, generally, subset) of space–time. Below we will have in mind just this, every time specifying the particular situation.

As we saw above, in the equivalence principle is involved a special class of coordinate systems or frames (of reference), usually called (local) inertial [2] or (local) Lorentz [1] in the physical literature and normal (and, by some authors, geodesic or Riemannian) in the mathematical one [14, Ch. V, Section 3] (see Appendix A). The main property of a frame of this class is that in it one can 'locally' neglect the effects of gravity (or of the accelerated motion of the frame), or, more strictly, that in it the gravitational field strength is 'locally' transformed to zero (or vanishes). Mathematically this is the corner-stone of the equivalence principle: if such frames do not exist it is meaningless, and conversely, if they exist it is meaningful, and the problem whether the equivalence principle is a principle (an axiom) or a theorem depends on the approach to the concrete theory under consideration (see Section 4).

In all of the gravitational theories known to the author the gravitational field strength is locally identified with the components of a certain linear connection, for instance with the Cristoffel symbols formed from the metric (Levi–Cevita's connection) in general relativity [1,2] or with the coefficients of the Riemann–Cartan connection in the U_4 theory [6]. Just this point connects physics with mathematics here and makes it possible the strict mathematical consideration of the above problem. In fact, in this context, the above special frames are coordinate systems (or local bases) in which the components of the corresponding linear connection locally vanish.

So, if locally the gravitational field strength is identified with the local components of a linear connection ∇, then it is meaningful to be spoken about the equivalence principle on some subset U of the space–time M if and only if in (a neighborhood of) U exist frames (coordinates, bases) in which the connection's components vanish on U. Thus there arises the mathematical problem for finding, if any, the linear connections ∇ on the set U and the coordinates, called *normal*, in a neighborhood of U in which the components of ∇ vanish on U. To the author are known the following basic results on this problem.

According to [14, Ch. V, Section 3] the existence of normal coordinates at a point ($U = \{x_0\}, x_0 \in M$) for symmetric linear connection has been proved at first in [15]. In 1922 Fermi [16] proved the existence of normal coordinates along any curve without self-intersections in the pseudo-Riemannian manifold of the general relativity. In many textbooks (see, e.g., [14,17]) it is proved that for symmetric linear connections normal coordinates exist in a neighborhood iff the connection is flat in it. The general case for symmetric linear connections is investigated in [18] where necessary and sufficient conditions for the existence of normal coordinates on submanifolds were found. All these results concern torsion free, i.e. symmetric, linear connections. In the cooresponding works it is also mentioned that for non-symmetric linear connections there are no normal coordinates (more precisely, holonomic normal coordinates do not exist). It seems that in [6], in fact without proof, the existence of anholonomic normal coordinates at a point for non-symmetric linear connections was mentioned for the first time. In 1992, in [11] and in [12] the existence of generally anholonomic local normal coordinates, called there special bases, was proved at a point and along a path, respectively, not only for any linear connection but also for arbitrary derivations of the tensor algebra over a differentiable manifold. The work [11] among others deals with the problem in a neighborhood: the sought for (anholonomic) normal coordinates exist only in the flat case (zero curvature of the derivation or connection). The paper [13] contains necessary and/or sufficient conditions for existence, holonomicity, and uniqueness of normal coordinates (special bases) on sufficiently general subsets of a differentiable manifold for arbitary derivations of the tensor algebra over it that, in particular, may be linear connections. In 1995 in [19] (independent of [11]) the existence of anholonomic normal coordinates (frames) at a point was proved for linear connections with torsion, a result which is a very special case of the ones of [13] or [11].

The cited results, some of which will be discussed in the next section, are the strict mathematical base for analyzing the equivalence principle.

3. On the general existence of normal coordinates

As we have said in Section 2, the problems connected with the existence (and uniqueness) or normal coordinates for symmetric linear connections were more or less completely investigated in [15,16,18]. In [10–13] analogous problems were studied in the case of arbitrary derivations of the tensor algebra over a differentiable manifold. In particular these derivations can be covariant differentiations (linear connections) with or without torsion. Thus, these works, a brief review of which is presented below, incorporate as their special cases the above cited ones concerning torsion free linear connections.

Any (S-) derivation of the tensor algebra over a manifold M is a map $D: X \mapsto D_X = L_X + S_X$, where X is a vector field, L_X is the Lie derivative along X, and S_X is (depending on X) tensor field of type $(1, 1)$ considered here as derivation [11,20].

If $\{E_i\}$ is a field of vector bases in the tangent to M bundle, then the *coefficients* $(W_X)^i_j$ of D are defined, e.g., through

$$D_X(E_j) = (W_X)^i_j E_i. \tag{3.1}$$

Here and below all Latin indices run from 1 to $\dim(M)$ and summation from 1 to $\dim(M)$ over repeated indices on different levels is assumed.

Let $W_X := [(W_X)^i_j]$ be the matrix formed from the coefficients $(W_X)^i_j$ of D. The change $\{E_i\} \to \{E'_i := A^j_i E_j\}$ of the basic vector fields induces

$$W_X \to W'_X = A^{-1}(W_X A + X(A)) \tag{3.2}$$

with $A := [A^i_j]$ and $X(A)$ being the action of X on A, i.e. $X(A) = [X(A^i_j] = [X^k E_k(A^i_j)]$.

From (3.1) or (3.2) it is evident that D is a covariant differentiation ∇ with (local) coefficients Γ^i_{jk} in $\{E_i\}$ iff $(W_X)^i_j = \Gamma^i_{jk}X^k$, i.e. if W_X depends linearly on X. In general, D is said to be *linear on (in)* $U \subseteq M$ *or along a map* $\eta : Q \to M$ for some set Q if in some basis (and hence in *all* bases) $\{E_i\}$ the relation $W_X(x) = \Gamma_k(x)X^k(x)$ is fulfilled for some matrix functions Γ_k and $x \in U$ or $x \in \eta(Q)$, respectively.

The (operators of) *curvature* R^D and *torsion* T^D of a derivation D are, respectively, $R^D(X, Y) := D_X \circ D_Y - D_Y \circ D_X - D_{[X,Y]}$ and $T^D(X, Y) := D_X Y - D_Y X - [X, Y]$ for any vector fields X and Y, $[X, Y]$ being their commutator.

Now the problem interesting for us has the following formulation. Let there be given a subset $U \subseteq M$. There have to be found all derivations D and the corresponding fields of bases $\{E_i\}$, defined on U or on a neighborhood of U, in which the components of D vanish on U, i.e. $W_X(x) = 0$ for $x \in U$. If such bases (frames) exist, we call them *normal bases* (resp. *normal frames*) for D (on U). Here and below we prefer to speak about normal bases (or frames) instead of normal coordinates because these bases (frames) are generally anholonomic, i.e. in the usual sense (holonomic or integrable) coordinates with the needed property do not exist and one has every time, when mentioning them, to add the appropriate adjective 'anholonomic' or 'holonomic'.

Now we shall present some basic results from [10–13] concerning the existence, uniqueness, and holonomicity of normal frames.

In neighborhoods the following results are valid [10,11]:

Proposition 3.1. *In a neighborhood* $U \subseteq M$ *there exists a normal frame for a derivation* D *iff it is a flat linear connection or iff it is flat* ($R^D = 0$) *and* $D_X|_{X=0} = 0$ *in* U.

Proposition 3.2. *The normal bases in* U *for* D, *if any, are connected by (homogeneous) linear transformations with constant coefficients and are holonomic (anholonomic) iff* $T^D = 0$ *(resp.* $T^D \neq 0$*) in* U.

Hence the flat (in U) linear connections are the only derivations for which there exist normal bases in neighborhoods. These frames are holonomic iff the connection is symmetric (torsion free).

At a given point our problem is solved by [10,11]:

Proposition 3.3. *At a point* $x_0 \in M$ *there exists a normal frame for a derivation* D *iff* D *is linear at* x_0.

Proposition 3.4. *The normal bases for* D, *at* x_0, *if any, are connected by linear transformations whose matrices vanish at* x_0 *under the action of the normal basic fields, and they are holonomic iff* D *is torsion free at* x_0.

As a linear connection is, evidently, a linear at (every) x_0 derivation, the last two propositions contain as a special case the hypothesis formulated in [6] as well as its strict formulation and proof in [19]: any linear connection admits normal frames at every fixed point which are holonomic iff it is symmetric.

Along an arbitrary path $\gamma : J \to M$, J being a real interval, the following propositions are fulfilled [12]:

Proposition 3.5. *Along γ (i.e. on $\gamma(J)$) there exists a normal basis for a derivation D iff D is linear along γ (i.e. on $\gamma(J)$).*

Proposition 3.6. *The normal along γ bases for D, if any, are connected through linear transformations whose matrices vanish along γ under the action of the normal basic fields. If they are holonomic, then D is torsion free on $\gamma(J)$ and, conversely, if D is torsion free on $\gamma(J)$ and there is a smooth normal basis along γ, then all of them are holonomic.*

As a linear connection ∇ is a derivation linear along any path, we see that any linear connection admits normal frames along every fixed path. If there is a holonomic basis for ∇ normal along γ, then ∇ is symmetric, and if ∇ is symmetric and there is a normal basis for it, smooth along γ, then all such bases are holonomic. In particular, for symmetric ∇ and paths without self-intersections we get in this way the classical result of [16].

If one is interested of derivations along paths (see the definition in [12, Section III]), there always exist holonomic, as well as anholonomic normal bases along any path γ. In particular, this is true for the covariant differentiation $\nabla_{\dot{\gamma}}$, along γ corresponding to a linear connection ∇ ($\dot{\gamma}$ is the tangent to γ vector field).

The general situation concerning normal bases is the following [13].

Proposition 3.7. *If on the set $U \subseteq M$ there exists a normal basis for a derivation D, then D is linear on U.*

But the opposite to this proposition is generally not valid (cf., e.g., Proposition 3.1).

Proposition 3.8. *In a set U the normal bases for D, if any, are connected by linear transformations whose matrices vanish on U under the action of these normal basic vector fields. If there is such a holonomic basis, then D is torsion free on U and, conversely, if D is torsion free on U and there is in U a smooth normal basis for D, then all normal in U bases for D are holonomic.*

Theorem 4 of [13] expresses a necessary and sufficient condition for existence of normal bases (frames) for linear derivations along maps with separable points of self-intersection. In particular it covers the case of arbitrary submanifolds of the space–time and the case of arbitrary linear connections, thus generalizing the results of [18]. Here we shall mention only the following corollary of this theorem. The zero- and one-dimensional cases are the only ones in which normal frame always exist for linear derivations on the corresponding sets (see resp. Propositions 3.3 and 3.5). In particular this is true for linear connections. On submanifolds of dimension $p = 2, \ldots, \dim M$ (for $\dim M \geq 2$) normal frames exist only as an exception in a case when some conditions are fulfilled (for $p = \dim M$, cf. Proposition 3.1).

4. The equivalence principle: Axiom or theorem?

It was shown in Section 2 that the equivalence principle is meaningless without a clear and strict understanding of what is a local inertial frame. Physically it can be defined as a frame in which the gravitational field strength (locally) vanishes. But then the question arises how this strength is described mathematically. In all (non-quantum) gravitational theories known to the author the gravitational field strength is (locally) identified with the components of some linear connection which leads to the identification of the class of inertial frames with the class of normal frames for this linear connection. Hence, in these theories the *physical concept 'inertial frame' coincides with the mathematical concept 'normal frame'*. in this way also automatically the problem of what 'local' (or 'locally') strictly means in the equivalence principle is solved: it simply means the set(s) on which the corresponding normal frame(s) is (are) defined.

The results of Section 3 imply that normal frames exist not only for linear connections but also for more general derivations (which are linear on the corresponding sets). So, the equivalence principle can be formulated for theories in which the gravitational field strength is identified with the components of certain derivation of the tensor algebra over the space–time. In this case one has to identify the inertial and normal frames too.

If one wants the normal frames to exist not only on a particular set (e.g. on a given path) but also on some class of subsets of the space–time (e.g. on all paths), then he again arrives to the case of linear connections if these subsets cover the whole space–time. (In the last case by Proposition 3.7 the derivation is linear at any space–time point which means that it is a linear connection.) Combining these results with Propositions 3.3 and 3.5 one derives:

Proposition 4.1. *The linear connections (covariant differentiations) are the only derivations for which normal bases exist at every space–time point or/and along every path in it.*

On other (families of) sets, even for linear connection, normal frames exist only as an exception (see, e.g., Proposition 3.1 and [13]).

Consequently, if one tries to formulate the equivalence principle he has to suppose that the gravitational field strength is identified with the cofficients of some linear connection. If this is done, then there exist local inertial frames ($\equiv$ normal frames).

Until now the 'first part' of the equivalence principle was discussed: it concerns inertial (normal) frames from mathematical point of view. Its 'second part' presupposes the existence of inertial frames and states that in them the "non-gravitational physical laws take their special relativity form". But here the question arises: when and in which frames the special relativity (and the physical laws in it) is (are) valid?

The answer is: in frames which are not accelerated or in which the gravitational field strength vanishes which, because of the empirical equality between inertial and gravitational masses, is one and the same thing [21]. Such frames are called, by definition, inertial too. This is not accidental because their class coincides with above-considered class of normal frames in which the gravitational field strength vanishes too. Hence, it turns out that by definition, empirically based on the equality of inertial and gravitational masses, the special relativity Nature laws are valid in the inertial frames.

So, what does the equivalence principle state in the end? The existence of inertial frames? No, because they are needed for its formulation and the fact of their existence is a consequence of the theory's mathematical background. Where are the special relativity laws valid? No, because this is either a question of definition: once the special relativity laws are established and experimentally checked, one has to extrapolate this fact by mathematically describing where they are valid. The above discussion shows that in this context *the equivalence principle asserts the coincidence of the two types of inertial frames: the normal frames, in which the components of a linear connection (or some other derivation) vanish, and the inertial frames, in which special relativity is valid.* But, as it was demonstrated above, this is a consequence of the fact that the gravitational field strength is mathematically described by the components of a certain linear connection. Thus, from this position, *equivalence principle is a theorem.*

It seems that for the first time such a conclusion was made in [3, Section 61] in the case of general relativity, where it is asserted that the equivalence principle "is contained in the hypotheses of the Riemannian character of space–time and mathematically is expressed in the possible introduction of local geodesic (i.e. normal – B.I.) coordinate systems along a time-like world line" [3, p. 307].

Can the equivalence principle be considered an axiom? Our opinion is that this is also possible, but not in its usual formulation(s) (see Section 2). For this purpose the 'equivalence principle' should be formulated as follows: *in any local frame of reference the gravitational field strength is described through identifying it with the local coefficients in this frame of a certain linear connection (or another derivation).* Implicitly in this statement the equality between the inertial and gravitational masses is incorporated which is supposed to be valid before the formulation of the usual equivalence principle, which in its turn, as was demonstrated above, is a consequence of it.

5. Can the equivalence principle be generalized?

In the usual formulation(s) of the equivalence principle the question for its generalization does not stand at all: it concerns a single theory (general relativity [1,2]) and its validity in other theories (such as the U_4 gravity theory [6]) was under question until recently. Our investigation shows that it is meaningful also in any gravitational theory based on linear connections. It is valid in such a theory at every point and along any path. On other subsets of the space–time it can be valid only as an exception. One can also formulate the equivalence principle in gravitational theories based on derivations more general than covariant differentiation. In such theories it can, in general, be valid on particular subsets of the space–time. If its validity in them is demanded on the whole space–time, then with necessity the corresponding derivation must be a covariant differentiation, i.e. one arrives again at a theory based on linear connections.

In sum, *the equivalence principle (in its usual formulation(s)) is valid in the whole space–time (at any point or along any path) in all gravitational theories based on linear connections.* (Note that the new formulation of the equivalence principle, presented in the end of the Section 4, serves just to select those theories.)

Further generalizations of the equivalence principle are possible in two directions: by generalizing the (mathematical) concept of 'normal' frame or by generalizing the description of the gravitational interaction (on the base different from the one of linear connections).

One possible such generalization is outlined in [22]. In it one supposes the tangent to the space–time bundle to be endowed with a linear transport along paths, which may not to be a parallel transport assigned to a linear connection. (For the general theory of such transports – see [23].) The gravitational field strength is then identified with the transport's coefficients. (The gravitational field itself can be described through the transport or its curvature.) Define the class of the normal frames to be the one of all bases (frames) in which the transport's coefficients vanish along an arbitrary given path. The so-defined normal frames always exist along any path or at any point (which is a degenerate path). In such a gravitational theory, which will be studied elsewhere, the equivalence principle is valid, for instance, in any of its formulations given is Section 2. Due to the equivalence established in [23] between linear transports along paths (generally in vector bundles) and derivations along paths, the sketched base for a possible gravitational theory can be formulated (equivalently) in terms of derivations along paths. Evidently, in such terms it is a straightforward generalization of the theories based on linear connections.

Another way for generalizing the equivalence principle is to extend the 'physical' area of its validity, i.e. to apply it to fields different from the gravitational one (cf. [24]). The reason for such possibility is the fact that the gauge (Yang–Mills) fields are from mathematical viewpoint linear connections (on vector bundles). This suggests the idea for such a formulation of the equivalence principle that it concerns all fields (interactions) described by means of gauge theories.

Appendix A. Normal, geodesic, Lorentz, and inertial frames

We called normal a special kind of local bases, frames, or coordinates investigated in the present paper. This needs some explanations.

For symmetric linear connections the local coordinates in which their components vanish at a given point are called normal in [14, Ch. V, Section 3] or in [1, Section 11.6]. In [20, Ch. III, Section 8] and in [5, p. 278] the local coordinates normal at a point, introduced there via the exponential map, for any linear connection (symmetric or not) are defined as such for which the symmetric part of the connection's components vanish at this point. Evidently, the latter definition includes the former one as a special case. Note that both the definitions originate from the consideration of the equation of geodesic lines [5,14,20]. This is the primary reason to call these local coordinates geodesic (or Riemannian, or normal Riemannian [1, Section 11.5]) in the special case of a Riemannian manifold [3, Section 42, p. 201], where they are (some times) equivalently introduced via the condition that in them the partial derivatives of the metric's components vanish at a given point [3, Section 42].

The case of a symmetric linear connection is investigated in [17, Ch. III, Section 7, pp. 156–158] (see the references therein too). A distinction between geodesic and normal at a point local coordinates has been made. Geodesic coordinates are called the ones in which at that point vanish the connection's components and normal coordinates are called the geodesic ones satisfying at the given point Eq. (7.23) of [17, Ch. III, Section 7] which, in particular, implies the vanishing at that point of the connection's components together with their symmetrized partial derivatives. (Note that the possibility for the existence of the last type of coordinates is ensured by our (non-) uniqueness result expressed by Proposition 3.4 with which is compatible the mentioned equation.) Analogous opinion is shared in [8, pp. 13–14].

It is known that the symmetric part of the connection symbols of arbitrary linear connection ∇ are directly connected with the equation of geodesic lines (curves, paths) and uniquely determine them [17,20]. By our opinion, this suggests the following convenient convention. Call normal or resp. geodesic on a set U a local coordinate system (basis, or frame), defined in a neighborhood of U, in which the local components of ∇ or resp. their symmetric pats vanish on U. Thus in the torsion free case the concepts of normal and geodesic coordinate system coincide. Generally a normal frame is geodesic, the converse being not valid. In this sense, the normal coordinates described in [17, p. 158] are a special type of (our) normal coordinates, specified by the additional conditions described in this reference consistent with Proposition 3.4. Note that the proposed definition is in accordance with the special one used in [19].

If one adopts the suggested convention, then the generalization from linear connections to arbitrary derivations D of the tensor algebra over a manifold is evident: only the concept of a normal frame is applicable because, generally, some symmetry properties of the coefficients of D cannot be spoken about. This explains the terminology accepted in the present paper.

Let us mention that the so-defined normal bases for D in U have a connection with a kind of generalized geodesic lines correspondig to D (cf. [25]) which will be discussed elsewhere.

In the physical literature, contrary to the mathematical one, there is a unique understanding what local inertial and Lorentz frames are. A local Lorentz coordinate system is defined for the (pseudo-)Riemannian space–time of general relativity as a one in which at a given point (or another set) the metric tensor coincides with the Minkowski metric tensor and all partial derivatives of the metrical components are zeros at this point (see, e.g., [1, Sections 8.5, 8.6, 13.6] or [9, Section 9.6]). (Note that this definition admits an evident generalization to arbitrary (pseudo-) Riemannian manifolds: only the Minkowski metric tensor has to be replaced with arbitrary fixed tensor.) A local inertial frame (of reference) at a given point (or another set) is defined as a one in which at this point (or an other set) the gravitational effects (or more precisely, the gravitational field strength) vanish (see [1, Sections 1.3, 1.6] or [9, Section 9.6]). When the gravitational field strength is identified with the local components of some linear connection, which is the usual situation [1,3,6,9], this means the vanishing of the connection's components at the given point. In general relativity this leads to the fact that any local Lorentz system is a local inertial frame [1, Section 13.3].

Thus, if the gravitational field strength is locally identified with the local components of some derivation D, then only the concept of a local inertial frame survives. Besides, if (maybe independently) a metric is presented then there also arises the class of local Lorentz frames; of such a type are the metric-affine gravitational theories. Generally, these types of frames, if both exist, need not be connected somehow with each other.

Appendix B. On derivations of the tensor algebra over a manifold

A derivation of the tensor algebra $\mathcal{T}(M)$ over a differentiable manifold M is a linear map $D : \mathcal{T}(M) \to \mathcal{T}(M)$ which satisfies the Leibnitz differentiation rule with respect to the tensor product, preserves the tensor's type, and commutes with the contractions of the tensor fields [20, Ch. I, Section 3]. By [20, Ch. I, Proposition 3.3] any D admits a unique representation in the form $D = L_X + S$ for some (unique for a given D) vector field X and tensor field S of type $(1, 1)$. Here S is considered a derivation of $\mathcal{T}(M)$ [20], which for a covariant differentiation ∇ is given through $S_X(Y) = \nabla_X(Y) - [X, Y]$, Y being a vector field.

Let $\{E_i, i = 1, \ldots, n := \dim(M)\}$ be a (coordinate or not [17]) local basis (frame) of vector fields in the tangent to M bundle. It is holonomic (anholonomic) if the vectors $E_1, \ldots, E_n$ commute (do not commute) [17]. Let T be a C^1 tensor field of type (p, q), p and q being integers or zero(s), with local components $T^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ with respect to the tensor basis associated with $\{E_i\}$. Here and below all Latin indices, maybe with some super- or subscripts, run from 1 to $n := \dim(M)$. Using the explicit action of L_X and S_X on tensor fields [20] and the usual summation rule about repeated indices on different levels we find the components of $D_X T$ to be

$$(D_X T)^{i_1 \cdots i_p}_{j_1 \cdots j_p} = X(T^{i_1 \cdots i_p}_{j_1 \cdots j_q}) + \sum_{a=1}^{p} (W_X)^{i_a}_{k} T^{i_1 \cdots i_{a-1} k i_{a+1} \cdots i_p}_{j_1 \cdots j_q}$$

$$- \sum_{b=1}^{q} (W_X)^{k}_{j_b} T^{i_1 \cdots i_p}_{j_1 \cdots j_{b-1} k j_{b+1} \cdots j_q}. \tag{B.1}$$

Here $X(f)$ denotes the action of $X = X^i E_i$ on the C^1 scalar function f, i.e. $X(f) = X^k E_k(f)$, and the explicit form of W_X (cf. (3.1)) is

$$(W_X)^i_j = (S_X)^i_j - E_j(X^i) + C^i_{kj} X^k, \tag{B.2}$$

where C^i_{kj} define the commutators of the basic vector fields by $[E_j, E_k] = C^i_{jk} E_i$.

From (B.2) or from (3.1) follows Eq. (3.2).

Using the equation $D_X = L_X + S_X$, one finds the followings representations for the curvature and torsion operators:

$$R^D(X,Y) = S_X \circ S_Y - S_Y \circ S_X + [X, S_Y\cdot] - [Y, S_X\cdot] + S_X([Y,\cdot]) - S_Y([X,\cdot]) - S_{[X,Y]},$$
$$T^D(X,Y) = S_X(Y) - S_Y(X) + [X,Y].$$

We have the local expressions:

$$[(R_D(X,Y))^i_j] = X(W_Y) - Y(W_X) + W_X W_Y - W_Y W_X - W_{[X,Y]}, \tag{B.3}$$
$$(T^D(X,Y))^i = (W_X)^i_j Y^j - (W_Y)^i_j X^j - C^i_{jk} X^j Y^k, \tag{B.4}$$

respectively. For a linear connection ∇ is fulfilled $(R^\nabla(X,Y))^i_j = R^i_{jkl} X^k Y^l$ and $(T^\nabla(X,Y))^i = T^i_{kl} X^k Y^l$ where R^i_{jkl} and T^i_{kl} are the components of the usual curvature and torsion tensors, respectively [17,20].

Other general results concerning derivations can be found in [20].

Appendix C. Sketch of some proofs

Propositions 3.1–3.8 are the strict mathematical basis for our analysis of the equivalence principle. Their full proofs can be found in [10–13]. Below are presented the main aspects of them.

Proof of Proposition 3.7. Let $\{E'_i = A^j_i E_j\}$ be a normal frame for D in U. Then $W'_X|_U = 0$ which by (3.2) is equivalent to $W_X(x) = \Gamma_k(x) X^k(x)$, $x \in U$ with $\Gamma_k = -(E_k(A))A^{-1}$, $A = [A^j_i]$. □

The first parts (necessity) of Propositions 3.1, 3.3 and 3.5 are corollaries from Proposition 3.7 when U is a neighborhood, or a point, or path, respectively. (Nore that in the first case $W_X = -(X(A))A^{-1}$ implies $R^D = 0$ due to (B.3).)

Proof of Proposition 3.1 (sufficiency). For a flat linear connection one can construct normal bases by fixing some basis at an arbitrary point and then transporting it to any point of U by means of the parallel transport generated by that connection. □

Proof of Proposition 3.3 (sufficiency). A local holonomic frame $\{E'_i = A^j_i \partial/\partial^j\}$ at a point x_0 can be constructed by choosing the coordinates $\{x^i\}$ such that $X = \partial/\partial x^1 (\neq 0$ at $x_0)$ and putting $A(z) = \mathbb{1} + C_k(x^k(z) - x^k(x_0))$, where $\mathbb{1}$ is the unit matrix and the matrices C_k are partially fixed through the conditions $(C_k)^i_j = (C_j)^i_k \in \mathbb{R}$ and $C_1 = W_X$. □

Proof of Propostion 3.5 (sufficiency). Let the path $\gamma : J \to M$ be without self-intersections and be contained in only one coordinate neighborhood. Let $V := J \times \cdots \times J$, where J is taken $n-1$ times. Let us fix a one-to-one C^1 map $\eta : J \times V \to M$ such that $\eta(\cdot, \mathbf{t}_0) = \gamma$ for some fixed $\mathbf{t}_0 \in V$, i.e. $\eta(s, \mathbf{t}_0) = \gamma(s)$, $s \in J$. (This is possible iff γ is without self-intersections.) In $U \cap \eta(J, V)$ we introduce coordinates $\{x^i\}$ by putting

$(x^1(\eta(s,\mathbf{t})),\ldots,x^n(\eta(s,\mathbf{t}))) = (s,\mathbf{t}), s\in J, \mathbf{t}\in V$. (This, again, is possible iff γ is without self-intersections.) Let $W_X(\gamma(s)) = \Gamma_k(\gamma(s))X^k(\gamma(s)), s\in J$. Then all normal along γ frames $\{E'_i = A^j_i\partial/\partial x^j\}$ are described by the matrix

$$\begin{aligned} A(\eta(s,\mathbf{t})) = &\left\{ \mathbb{1} - \sum_{k=2}^{n} \Gamma_k(\gamma(s))[x^k(\eta(s,\mathbf{t})) - x^k(\eta(s,\mathbf{t}_0))] \right\} \\ &\times Y(s,s_0; -\Gamma_1\circ\gamma)B(s_0,\mathbf{t}_0;\eta) \\ &+ B_{kl}(s,\mathbf{t};\eta)[x^k(\eta(s,\mathbf{t})) - x^k(\eta(s,\mathbf{t}_0))] \\ &\times [x^l(\eta(s,\mathbf{t})) - x^l(\eta(s,\mathbf{t}_0))]. \end{aligned} \tag{C.1}$$

Here $\mathbb{1}$ is the unit matrix, $s_0\in J$ is fixed, B is any non-degenerate matrix function of its arguments, the matrix functions B_{kl} are such that they and their first derivatives are bounded when $\mathbf{t}\to\mathbf{t}_0$, and $Y = Y(s,s_0;Z)$, with Z being a continous matrix function of s, is the unique solution of the matrix initial-value problem [26, Ch. IV, Section 1]

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = ZY, \qquad Y|_{s=s_0} = \mathbb{1}, \qquad Y = Y(s,s_0;Z).$$

In the case when γ has self-intersections and/or is not contained in only one coordinate neighborhood the frames normal along γ are constructed from the ones for the pieces of γ satisfying the conditions at the beginning of this proof. □

Proof of Proposition 3.8 (first part). If $\{E_i\}$ and $\{E'_i = A^j_iE_j\}$ are normal in U, then $W_X|_U = W'_X|_U = 0$, which by (3.2) means that $X(A)|_U = 0$, i.e. $E_i(A)|_U = 0$ as X is arbitrary. Conversely, if $\{E_i\}$ is normal in U, i.e. $W_X|_U = 0$, and $E'_i = A^j_iE_j$ with $E_i(A)|_U = 0$, then, again by (3.2), we get $W'_X|_U = 0$, i.e. $\{E'_i\}$ is normal in U. □

If we specify U to be neighborhood, or a point, or a curve (i.e. the set $\gamma(J)$), then from the first part of Proposition 3.8 follow the first parts of Propositions 3.2, 3.4, and 3.6, respectively. Analogously, their second parts are corollaries from the second part of Proposition 3.8.

Proof of Proposition 3.8 (second part). If $\{E'_i\}$ is a normal frame in U, then $W'_X|_U = 0$ which, due to (B.4), implies $T^D(E'_i,E'_j)|_U = -[E'_i,E'_j]|_U$. So, the holonomicity condition $[(E'_i,E'_j]|_U = 0$ is equivalent to $T^D|_U = 0$. □

The considered propositions can be proved also independently, which is done in the above-cited references, where other details and results can be found.

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